Fermion Doubling Theorems in Two-Dimensional Non-Hermitian Systems for Fermi Points and Exceptional Points

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The fermion doubling theorem plays a pivotal role in Hermitian topological materials. It states, for example, that Weyl points must come in pairs in three-dimensional semimetals. Here, we present an extension of the doubling theorem to non-Hermitian lattice Hamiltonians. We focus on two-dimensional non-Hermitian systems without any symmetry constraints, which can host two different types of topological point nodes, namely, (i) Fermi points and (ii) exceptional points. We show that these two types of protected point nodes obey doubling theorems, which require that the point nodes come in pairs. To prove the doubling theorem for exceptional points, we introduce a generalized winding number invariant, which we call the "discriminant number." Importantly, this invariant is applicable to any two-dimensional non-Hermitian Hamiltonian with exceptional points of arbitrary order and, moreover, can also be used to characterize nondefective degeneracy points. Furthermore, we show that a surface of a three-dimensional system can violate the non-Hermitian doubling theorems, which implies unusual bulk physics.

Introduction.—Fermion doubling theorems [1–3] are an important concept in the topological band theory of condensed matter physics [4–7]. They state that topological point nodes in the energy spectrum of lattice Hamiltonians must come in pairs. Thereby they prevent the occurrence of quantum anomalies in lattices. This is because, for a single point node, the low-energy physics is described by a field theory with a quantum anomaly, while for two point nodes the anomalies cancel. Well-known examples of doubled point nodes include the two Dirac points of graphene [8] and the two Weyl points of magnetic Weyl semimetals [6,9]. While the doubling theorems must be fulfilled in the bulk of any lattice Hamiltonian, they may be violated on a lattice surface. For example, the topological insulator with time-reversal symmetry exhibits a single Dirac point with parity anomaly on its surface [10]. These anomalous surface states lead to unusual physical responses and give a powerful diagnostic of the nontrivial bulk topology [11,12].

Recently, topological band theory has been extended to non-Hermitian Hamiltonians [13–64], which can be realized in, e.g., photonic cavity arrays [65–68], and provide effective descriptions of open quantum systems [15–20, 38,69,70], where energy is not conserved due to, e.g., dissipation or particle gain and loss. In contrast to the Hermitian case, two-dimensional (2D) non-Hermitian Hamiltonians can exhibit three different types of point nodes, namely (i) Fermi points (FPs), (ii) exceptional points (EPs), and (iii) nondefective degeneracy points (NDPs). Both at an EP and a NDP two (or more) energy bands become degenerate at a degeneracy point (DP). However, at an EP the corresponding eigenstates coalesce [71] (become identical), while at a NDP the eigenstates remain distinct. For this reason, non-Hermitian Hamiltonians at EPs are nondiagonalizable and can only be reduced to Jordan block forms [71]. In the absence of symmetry, both FPs and EPs can be topologically stable in 2D, meaning that these point nodes cannot be removed by perturbations [30,34,45,50].

The physics of EPs has recently attracted attention [33,72–97], in particular, in the context of photonic platforms, where they have many interesting applications, for example, as optical omnipolarizers [98] or as sensors with enhanced sensitivity [77,87]. While the occurrence and stability of FPs and EPs have been studied in various settings, the existence of doubling theorems for these topological point nodes remains unknown. In this Letter, we derive doubling theorems for FPs and EPs in 2D periodic lattice Hamiltonians without any symmetry constraints. For that purpose, we consider a non-Hermitian Hamiltonian $\mathcal{H}(\boldsymbol{k})$ possessing well-separated complex energy bands [30,34], except for some possible DPs, which are either EPs or NDPs. The proof of the doubling theorems

relies on the fact that topological points carry nonzero topological charges, whose sum must vanish in the entire Brillouin zone (BZ) [1–3] due to its periodicity, i.e.,

$$\sum_{\boldsymbol{k}_i \in \mathrm{BZ}} C(\boldsymbol{k}_i) = 0, \tag{1}$$

where $C(\boldsymbol{k}_i)$ is the topological charge of a topological point located at $\boldsymbol{k}_i$ in the BZ. Particularly, for 2D non-Hermitian systems, topological points can be FPs or EPs. The charges $C(\boldsymbol{k}_i)$ are defined in terms of an integral of some topological charge density, along a closed contour that counterclockwise encircles the FP or EP. Equation (1) then follows by continuously deforming the integration contours to the boundary of the BZ. We find that for FPs the appropriate charge density is the logarithmic derivative of $\det[\mu - \mathcal{H}(\boldsymbol{k})]$, while for EPs it is the logarithmic derivative of the *discriminant* of $\mathcal{H}(\boldsymbol{k})$ with respect to the energy E. We demonstrate by several examples that EPs do not need to be branch point singularities, contrary to previous reports [19]. Finally, we show that the doubling theorems can be violated at surfaces in 3D systems, which implies unusual properties of the bulk bands. In particular, inversion-symmetric or reflection-symmetric systems can host single FPs or EPs on their surfaces, which, however, must be accompanied by Fermi lines or exceptional lines in the bulk.

Doubling theorem for FPs.—We start with the doubling theorem for FPs of generic 2D non-Hermitian Hamiltonians $\mathcal{H}(\boldsymbol{k})$ with complex energy bands $E_i(\boldsymbol{k})$. The FPs of $\mathcal{H}(\boldsymbol{k})$ are defined as those points $\boldsymbol{k}_F^j$ in the BZ, where the complex chemical potential μ intersects with one of the energy bands $E_i(\boldsymbol{k})$, i.e., $\mu - E_i(\boldsymbol{k}_F^j) = 0$ for some i. By choosing a proper basis, each entry of $\mathcal{H}(\boldsymbol{k})$ is single valued in the entire BZ. The location of the FPs can then be obtained from the characteristic polynomial of $\mathcal{H}(\boldsymbol{k})$

$$f_\mu(\boldsymbol{k}) \equiv \det[\mu - \mathcal{H}(\boldsymbol{k})] = \prod_i [\mu - E_i(\boldsymbol{k})]. \tag{2}$$

That is, the FPs are located at those $\boldsymbol{k}_F^j$ where $f_\mu(\boldsymbol{k}_F^j) = 0$. Since $f_\mu(\boldsymbol{k})$ is a complex function, this gives the two

conditions [$\mathrm{Re}f_\mu(\boldsymbol{k}_F^j) = 0$ and $\mathrm{Im}f_\mu(\boldsymbol{k}_F^j) = 0$], whose solutions yield two line loops in the 2D BZ, see Fig. 1(a1). The crossings of these two loops give the positions of the FPs. Pictorially, we can see that the two loops must cross each other an even number of times, thereby suggesting a doubling theorem for FPs [99]. Moreover, we observe from Fig. 1(a2) that small perturbations only shift the paths of the loops, but do not remove the FPs.

These observations can be made more precise using the mathematical formalism of topological invariants. For this purpose we define the global winding number invariant [44]

$$W(\boldsymbol{k}_F^j) = \frac{i}{2\pi} \oint_{\Gamma(\boldsymbol{k}_F^j)} d\boldsymbol{k} \cdot \nabla_{\boldsymbol{k}} \ln f_\mu(\boldsymbol{k}), \tag{3}$$

where the integration path $\Gamma(\boldsymbol{k}_F^j)$ is a loop encircling $\boldsymbol{k}_F^j$ counterclockwise, as shown in Fig. 1(b1). Since $f_\mu(\boldsymbol{k})$ is single valued in the entire BZ, the winding number $W(\boldsymbol{k}_F^j)$ is quantized to an integer, which endows the FP at $\boldsymbol{k}_F^j$ with a topological charge. If the winding number is nonzero, the integration path $\Gamma(\boldsymbol{k}_F^j)$ in Eq. (3) cannot be smoothly shrunk to a single point, due to the presence of a singularity at the FP. This guarantees the topological stability of the FP and protects it against gap opening, even in the presence of perturbations. To derive the doubling theorem for FPs, we sum over the winding numbers of all FPs in the BZ

$$\sum_{\boldsymbol{k}_F^j \in \mathrm{BZ}} W(\boldsymbol{k}_F^j) = \frac{i}{2\pi} \oint_{\partial \mathrm{BZ}} d\boldsymbol{k} \cdot \nabla_{\boldsymbol{k}} \ln f_\mu(\boldsymbol{k}) = 0. \tag{4}$$

The above sum must vanish, because the integration paths of Eq. (3) can be continuously deformed to the BZ boundary $\partial \mathrm{BZ}$, as the FPs are the only singularities in the integrand [see Fig. 1(b)]. Hence, each FP with a positive topological charge must be accompanied by a FP with negative topological charge. This proves the doubling theorem (1) for FPs in 2D non-Hermitian systems.

An example of doubled FPs.—We use an example to demonstrate the topological properties of the FPs by studying a two-band Hamiltonian, given by
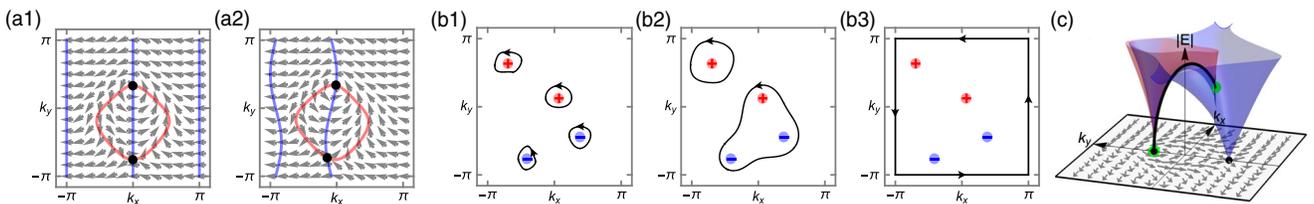
FIG. 1. (a) In the 2D BZ, Fermi points (black points) are located at the crossings of the red line [$\mathrm{Re}f_\mu(\boldsymbol{k}) = 0$] and the blue line [$\mathrm{Im}f_\mu(\boldsymbol{k}) = 0$]. The gray arrows display the orientation of the vector field [$\mathrm{Re}f_\mu(\boldsymbol{k}), \mathrm{Im}f_\mu(\boldsymbol{k})$]. (a2) Shows that a small perturbation does not destroy the Fermi points. (b) The integration paths $\Gamma(\boldsymbol{k}_F^j)$ (b1) of Eq. (3) [or Eq. (7)] can be continuously deformed to the boundary of the BZ (b3), without passing through any singular point (b2). The blue (red) dots represent Fermi points, or degeneracy points, with positive (negative) topological charge. (c) The absolute value of the complex energy bands $|E(\boldsymbol{k})|$ (red and blue surfaces) and vector field of the characteristic polynomial [$\mathrm{Re}f_\mu(\boldsymbol{k}), \mathrm{Im}f_\mu(\boldsymbol{k})$] (gray arrows) for the two-band model (5). The black and green points are Fermi points and exceptional points, respectively. The black arc represents the branch cut that connects the two exceptional points.

$$\mathcal{H}(\boldsymbol{k}) = h_0(\boldsymbol{k})\sigma_0 + \boldsymbol{h}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}, \qquad (5)$$

where $h_0(\boldsymbol{k}) = \frac{1}{2}\sin k_y$, $h_x(\boldsymbol{k}) = \sin k_x - i$, $h_y(\boldsymbol{k}) = \sin k_y$, and $h_z(\boldsymbol{k}) = \cos k_x + \cos k_y - 2$. With chemical potential $\mu = \sqrt{3}/4$, this example has two FPs located at $\boldsymbol{k}_F^- = (0, -0.479\pi)$ and $\boldsymbol{k}_F^+ = (0, \pi/3)$ with winding numbers $W(\boldsymbol{k}_F^{\pm}) = \mp 1$, such that the doubling theorem is satisfied [Fig. 1(c)]. The energy spectrum of $\mathcal{H}(\boldsymbol{k})$ is given by $E_{\pm}(\boldsymbol{k}) = (s_y/2) \pm \sqrt{5 - 4(c_x + c_y) + c_{x+y} + c_{x-y} - 2is_x}$, where $c_{x/y} = \cos k_{x/y}$, $s_{x/y} = \sin k_{x/y}$, and $c_{x\pm y} = \cos(k_x \pm k_y)$. This spectrum is multivalued and exhibits a branch cut that is terminated by two EPs, located at $(0, \pm\pi/3)$ with energies $E - \mu = 0$ and $E - \mu = -\sqrt{3}/2$, respectively. The fact that one of the EPs coincides with one of the FPs is purely accidental. In general, EPs and FPs of generic non-Hermitian Hamiltonians are at different positions.

Discriminant and DPs.—Next, we turn to DPs of generic non-Hermitian Hamiltonians and explain how they can be found in an efficient manner using the discriminant of the characteristic polynomial $f_E(\boldsymbol{k})$. Here, $f_E(\boldsymbol{k})$ is defined as in Eq. (2), but the chemical potential μ is replaced by the energy E. A DP occurs at $\boldsymbol{k}_D$ when $E_i(\boldsymbol{k}_D) = E_j(\boldsymbol{k}_D)$ for some $i \neq j$. Hence, the polynomial $f_E(\boldsymbol{k})$ must have a double (or multiple) root at $\boldsymbol{k}_D$. Moreover, the discriminant of $f_E(\boldsymbol{k})$, which is defined as

$$\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k}) = \prod_{i<j}[E_i(\boldsymbol{k}) - E_j(\boldsymbol{k})]^2, \qquad (6)$$

must vanish at $\boldsymbol{k}_D$; i.e., there is a DP at $\boldsymbol{k}_D$, if and only if $\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k}_D) = 0$. Apparently, the DPs are computed more efficiently from the zeros of the discriminant, rather than by explicitly calculating all roots of $f_E(\boldsymbol{k})$. This is because the discriminant can be computed directly from the determinant of the Sylvester matrix of $f_E(\boldsymbol{k})$ and $\partial_E f_E(\boldsymbol{k})$, see Supplemental Material [100]. Hence, determining the zeros of $\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})$ is an efficient way to find all DPs in the entire BZ at any energy [101].

The discriminant has the additional advantage of being single valued, since the coefficients of $f_E(\boldsymbol{k})$ are single valued [102]. This property is key to define a *quantized* invariant in terms of $\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})$ and to prove the doubling theorem of DPs. Before doing so, let us first give an illustrative argument for why DPs must satisfy a doubling theorem. We note that zeros of the discriminant must satisfy the two constraints $\mathrm{Re}[\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})] = 0$ and $\mathrm{Im}[\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})] = 0$. The solutions to these two equations are two line loops in the 2D BZ, whose crossings give the positions of the DPs. Since two loops in a periodic BZ generally cross each other an even number of times, DPs must come in pairs and satisfy a doubling theorem [103].

Finally, we remark that, in the absence of extra symmetries, DPs have, in general, only twofold degeneracy.

That is, any DP with higher degeneracy can be split into multiple twofold degenerate DPs by an infinitesimally small perturbation [100]. For this reason we focus only on twofold degenerate DPs.

Doubling theorem for DPs.—The invariant that characterizes the topology of DPs, which can be EPs or NDPs, is given in terms of a contour integral over the discriminant

$$\nu(\boldsymbol{k}_D^l) = \frac{i}{2\pi} \oint_{\Gamma(\boldsymbol{k}_D^l)} d\boldsymbol{k} \cdot \nabla_{\boldsymbol{k}} \ln \mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k}), \qquad (7)$$

where $\Gamma(\boldsymbol{k}_D^l)$ is a loop counterclockwise encircling the DP at $\boldsymbol{k}_D^l$. Since $\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})$ is single valued, this invariant is a quantized winding number, which we call the discriminant number. The mathematical structure is almost identical to the winding number (3) characterizing FPs. The only difference is that $\det[\mu - \mathcal{H}(\boldsymbol{k})]$ in the integrand of Eq. (3) is replaced by $\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})$. A nonzero quantized value of ν guarantees the stability of DPs against gap opening. Using Eq. (7) we can identify, in a computationally efficient manner, any stable DP between any pair of bands [100]. To obtain the doubling theorem for DPs, we sum over the discriminant numbers of all DPs in the BZ

$$\sum_{\boldsymbol{k}_D^l \in \mathrm{BZ}} \nu(\boldsymbol{k}_D^l) = \frac{i}{2\pi} \oint_{\partial\mathrm{BZ}} d\boldsymbol{k} \cdot \nabla_{\boldsymbol{k}} \ln \mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k}) = 0. \qquad (8)$$

Since the DPs are the only singularities in the integrand of Eq. (7), the integration paths in this sum can be continuously deformed to the BZ boundary $\partial\mathrm{BZ}$, see Fig. 1(b). Hence, the above sum must vanish and, therefore, the discriminant number of the DPs must cancel pairwise. This proves the doubling theorem (1) for DPs.

We note that the topology of DPs formed by two bands has previously been characterized using the vorticity invariant [34,50]

$$\nu_{ij}(\boldsymbol{k}_D^l) = -\frac{1}{2\pi} \oint_{\Gamma(\boldsymbol{k}_D^l)} \nabla_{\boldsymbol{k}} \arg[E_i(\boldsymbol{k}) - E_j(\boldsymbol{k})] \cdot d\boldsymbol{k}, \qquad (9)$$

where i and j label the two bands and $\arg(z) = -i \ln(z/|z|)$. We find, after some algebra [100], that the discriminant number is equal to the vorticity invariant summed over all pairs of distinct bands, i.e., $\nu(\boldsymbol{k}_D^l) = \sum_{i \neq j} \nu_{ij}(\boldsymbol{k}_D^l)$.

Examples of doubled DPs.—We use two examples to illustrate the doubling theorem for DPs and to show that only EPs are generically stable, while NDPs can be deformed to EPs by arbitrarily small perturbations.

The first example contains two NDPs and is given by the two-band Hamiltonian

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} 0 & F(\boldsymbol{k}) \\ G(\boldsymbol{k}) & 0 \end{pmatrix}, \qquad (10)$$
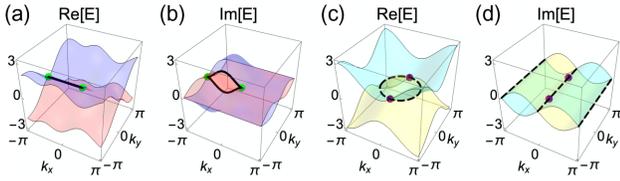
FIG. 2. The complex energy spectra for the DP examples. (a), (b) Shows that the Hamiltonian (10) possesses two NDPs, which are the ends of the branch cut (solid lines). (c),(d) Shows that in the Hamiltonian (11) an EP and a NDP do not connect to any branch cut, although these points are located at the crossing of the energy bands (dashed lines).

where $F(\mathbf{k}) = \sin^2 k_x - \frac{1}{2}\sin^2 k_y + 2i \sin k_x \sin k_y + \cos k_y - 1$ and $G(\mathbf{k}) = \sin k_x - i \sin k_y + \cos k_y - 1$. The energy spectrum is $E_\pm(\mathbf{k}) = \pm\sqrt{F(\mathbf{k})G(\mathbf{k})}$ and the characteristic polynomial reads $f_E(\mathbf{k}) = E^2 - F(\mathbf{k})G(\mathbf{k})$. From this we obtain the discriminant $\text{Disc}_E[\mathcal{H}](\mathbf{k}) = 4F(\mathbf{k})G(\mathbf{k})$, which has zeros at $(0,0)$ and $(\pi,0)$ corresponding to two NDPs [104]. Using Eq. (7), we find that these two NDPs have $\nu = \pm 1$, thereby satisfying the doubling theorem. These two NDPs are end points of a branch cut, demonstrating that branch cuts do not need to be terminated by EPs as shown in Figs. 2(a) and 2(b). However, this is an unstable situation, since the infinitesimally small perturbation $\delta\sigma_z$ turns the NDPs into EPs.

The second example contains one NDP and one EP and is described by the following Hamiltonian:

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} A(\mathbf{k}) & B(\mathbf{k}) \\ 0 & -A(\mathbf{k}) \end{pmatrix}, \qquad (11)$$

where $A(\mathbf{k})=1-\cos k_x-\cos k_y+i\sin k_x$ and $B(\mathbf{k})=1-\sin k_y$. The spectrum is $E_\pm(\mathbf{k}) = \pm A(\mathbf{k})$ and the characteristic polynomial reads $f_E(\mathbf{k}) = E^2 - A^2(\mathbf{k})$, from which we obtain the discriminant $\text{Disc}_E[\mathcal{H}](\mathbf{k}) = 4A^2(\mathbf{k})$. Solving for the zeros of the discriminant, we find two DPs located at $(0, \pm\pi/2)$. The DP at $(0, -\pi/2)$ is an EP with discriminant number $\nu = -2$, while the DP at $(0, +\pi/2)$ is a NDP with $\nu = +2$, such that the doubling theorem is satisfied. We observe that the EP does not terminate a branch cut as shown in Figs. 2(c) and 2(d), since the spectrum $E_\pm(\mathbf{k})$ is single valued in the entire BZ. But this is a fine-tuned situation, which is destabilized by the infinitesimally small deformation $\eta\sigma_x$. This perturbation splits the NDP and the EP each into two EPs with $\nu = \pm 1$, which become end points of branch cuts.

In general, a twofold degenerate EP with $\nu = \pm 1$ is always a branch cut termination of the energy spectrum (see the proof in the Supplemental Material [100] and cf. [19]). From the insights gained by the above examples, we prove in the Supplemental Material [100] that in two dimensions NDPs are unstable; i.e., they can be deformed into EPs by generic perturbations. Furthermore, EPs with $|\nu| > 1$ are

split into several EPs with $|\nu| = 1$ by small perturbations and, importantly, only EPs with $\nu = \pm 1$ are stable.

Doubling theorem for EPs.—Taking together the above results, we conclude that in two dimensions the only stable DPs are EPs with $\nu = \pm 1$. Allowing for generic perturbations, these EPs must come in pairs with opposite discriminant number ν.

Anomalous FPs and EPs at surfaces.—We close this Letter by discussing anomalous FPs and EPs (or DPs) at surfaces, which violate the doubling theorems. Surfaces of 3D systems can be viewed, in a sense, as one half of 2D bulk systems. As a consequence, surfaces can host, in principle, an odd number of stable FPs or EPs, thereby breaking the doubling theorems. In the presence of these anomalous surfaces, the bulk exhibits unusual topological properties, which depend on the crystalline symmetries. In this regard, we separately discuss the anomalous physics with and without the symmetries.

(i) First, we consider a symmetry that relates the top and bottom surfaces of a given 3D system. For concreteness, let us consider two surfaces that are related by reflection or inversion, which act on the two surface Hamiltonians as

$$P_\pm \mathcal{H}_{\text{top}}(k_x, k_y)P_\pm^{-1} = \mathcal{H}_{\text{bot}}(\pm k_x, \pm k_y), \qquad (12)$$

where $P_\pm$ is a unitary operator implementing reflection $(+)$ or inversion $(-)$. Focusing on surface EPs, we now let symmetry (12) act on the discriminant number ν, Eq. (7). From this we find that ν summed over all EPs at the top surface is equal to ν summed over all EPs at the bottom surface, i.e.,

$$\sum_{\mathbf{k}_D^l \in \text{BZ}_{\text{top}}} \nu(\mathbf{k}_D^l) = \sum_{\mathbf{k}_D^l \in \text{BZ}_{\text{bot}}} \nu(\mathbf{k}_D^l). \qquad (13)$$

Hence, as opposed to 3D topological insulators, the topological charges of the EPs on the top and bottom surfaces do not cancel. Therefore, there must exist additional EPs with nonzero ν in the 3D bulk that compensate the nonvanishing sum of the discriminant numbers on the two surfaces. In fact, since nonzero ν is defined in terms of a line integral, which cannot be deformed to vanish, there must exist entire lines of EPs in the 3D bulk [105]. Therefore, if there are surface EPs violating the doubling theorem, the 3D bulk must contain at least one exceptional line topologically protected by the nonzero discriminant number. Hence, the entire system contains two exotic non-Hermitian physics phenomena—anomalous surfaces and exceptional lines in the bulk [45–50].

By a similar derivation, we can prove the same property also for surface FPs. That is, if there are surface FPs violating the doubling theorem in an inversion or reflection-symmetric system, the bulk must be gapless (no point gap at the FP energy level) and must have at least one Fermi line. In the Supplemental Material [100] we present an

example of a 3D non-Hermitian lattice model, which exhibits these anomalous surface EPs and FPs together with bulk Fermi lines and exceptional lines.

(ii) Second, we consider a 3D Hamiltonian without FPs and EPs in the bulk, which breaks both reflection and inversion symmetry. We find that the surface of such a system can still violate the non-Hermitian doubling theorem [106]. In the Supplemental Material [100], we provide an example of a 3D Hamiltonian with a bulk point gap and surfaces that violate the FP doubling theorem. Since the top and bottom surfaces have opposite topological charges, the FP doubling theorem is only violated for an individual surface but is satisfied for the entire 3D system. (This is similar to Hermitian topological gapped systems, for example, surfaces of 3D topological insulators, which exhibit single Dirac points, both at the top and the bottom surfaces, with opposite topological charges [10].) Gapped bulk systems with such anomalous surfaces exhibit a nontrivial non-Hermitian topology, which is described by the 3D winding number [22,55,57,106]. Thus, the breaking of the non-Hermitian doubling theorem at the surface allows one to identify the nontrivial topology in the bulk.

Conclusion.—In summary, we have derived doubling theorems for FPs and EPs in generic 2D non-Hermitian lattice Hamiltonians. To derive the doubling theorem for exceptional points, we have introduced a new topological invariant, which we call the discriminant number. This discriminant number endows EPs with a quantized topological charge. The doubling theorem ensures that a single EP of first order must be accompanied by another first-order EP with opposite topological charge as all of the energy bands are taken into account. We have shown that 2D non-Hermitian Hamiltonians with fine-tuned parameters can also exhibit higher-order EPs or nondefective degeneracy points. These are, however, unstable and can be removed by arbitrarily small perturbations. Furthermore, we have clarified the relation between EPs and branch cuts, namely, only twofold degenerate EPs of first order must necessarily be branch cut terminations. Finally, we studied anomalous surfaces of 3D bulk systems that violate the non-Hermitian doubling theorems. In the presence of inversion or reflection symmetry, this violation leads to the emergence of topologically protected Fermi or exceptional lines in the bulk. In the absence of these symmetries, the bulk is gapped and must exhibit nontrivial topology.

Note added.—Recently, we became aware of a related preprint [107], which makes use of the discriminant to classify non-Hermitian Hamiltonians. We also noticed a relevant study [106], which first proposed a non-Hermitian model possessing a point gap and surfaces violating the FP doubling theorem.

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[1] H. Nielsen and M. Ninomiya, Phys. Lett. **105B**, 219 (1981).

[2] H. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981).

[3] H. Nielsen and M. Ninomiya, Nucl. Phys. **B193**, 173 (1981).

[4] C. Fang and L. Fu, Sci. Adv. **5**, eaat2374 (2019).

[5] B. Béri, Phys. Rev. B **81**, 134515 (2010).

[6] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).

[7] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).

[8] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).

[9] A. A. Burkov and L. Balents, Phys. Rev. Lett. **107**, 127205 (2011).

[10] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).

[11] S. Ryu, J. E. Moore, and A. W. W. Ludwig, Phys. Rev. B **85**, 045104 (2012).

[12] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).

[13] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).

[14] N. Moiseyev, *Non-Hermitian Quantum Mechanics* (Cambridge University Press, Cambridge, 2011).

[15] I. Rotter and J. P. Bird, Rep. Prog. Phys. **78**, 114001 (2015).

[16] L. Feng, R. El-Ganainy, and L. Ge, Nat. Photonics **11**, 752 (2017).

[17] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Nat. Phys. **14**, 11 (2018).

[18] S. K. Ozdemir, S. Rotter, F. Nori, and L. Yang, Nat. Mater. **18**, 783 (2019).

[19] M.-A. Miri and A. Alù, Science **363**, eaar7709 (2019).

[20] S. K. Gupta, Y. Zou, X.-Y. Zhu, M.-H. Lu, L.-J. Zhang, X.-P. Liu, and Y.-F. Chen, Adv. Mater. **32**, 1903639 (2020).

[21] V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres, Eur. Phys. J. Spec. Top. **227**, 1295 (2018).

[22] A. Ghatak and T. Das, J. Phys. Condens. Matter **31**, 263001 (2019).

[23] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[24] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).

[25] M. A. Stephanov, Phys. Rev. Lett. **76**, 4472 (1996).

[26] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77,** 570 (1996).

[27] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102,** 065703 (2009).

[28] S. Longhi, Phys. Rev. Lett. **105,** 013903 (2010).

[29] T. E. Lee, Phys. Rev. Lett. **116,** 133903 (2016).

[30] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. **118,** 040401 (2017).

[31] R. A. Molina and J. González, Phys. Rev. Lett. **120,** 146601 (2018).

[32] V. Kozii and L. Fu, arXiv:1708.05841.

[33] M. Papaj, H. Isobe, and L. Fu, Phys. Rev. B **99,** 201107(R) (2019).

[34] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120,** 146402 (2018).

[35] H. Shen and L. Fu, Phys. Rev. Lett. **121,** 026403 (2018).

[36] S. Yao and Z. Wang, Phys. Rev. Lett. **121,** 086803 (2018).

[37] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. **121,** 136802 (2018).

[38] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123,** 170401 (2019).

[39] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123,** 246801 (2019).

[40] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121,** 026808 (2018).

[41] K. Yokomizo and S. Murakami, Phys. Rev. Lett. **123,** 066404 (2019).

[42] Z. Yang, K. Zhang, C. Fang, and J. Hu, Phys. Rev. Lett. **125,** 226402 (2020).

[43] K. Zhang, Z. Yang, and C. Fang, Phys. Rev. Lett. **125,** 126402 (2020).

[44] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. **124,** 086801 (2020).

[45] Y. Xu, S.-T. Wang, and L.-M. Duan, Phys. Rev. Lett. **118,** 045701 (2017).

[46] A. Cerjan, M. Xiao, L. Yuan, and S. Fan, Phys. Rev. B **97,** 075128 (2018).

[47] J. Carlström and E. J. Bergholtz, Phys. Rev. A **98,** 042114 (2018).

[48] Z. Yang and J. Hu, Phys. Rev. B **99,** 081102(R) (2019).

[49] Z. Yang, C.-K. Chiu, C. Fang, and J. Hu, Phys. Rev. Lett. **124,** 186402 (2020).

[50] K. Kawabata, T. Bessho, and M. Sato, Phys. Rev. Lett. **123,** 066405 (2019).

[51] T. Liu, Y.-R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. **122,** 076801 (2019).

[52] Z. Zhang, M. Rosendo López, Y. Cheng, X. Liu, and J. Christensen, Phys. Rev. Lett. **122,** 195501 (2019).

[53] X.-W. Luo and C. Zhang, Phys. Rev. Lett. **123,** 073601 (2019).

[54] C. H. Lee, L. Li, and J. Gong, Phys. Rev. Lett. **123,** 016805 (2019).

[55] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8,** 031079 (2018).

[56] H. Zhou and J. Y. Lee, Phys. Rev. B **99,** 235112 (2019).

[57] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X **9,** 041015 (2019).

[58] C.-H. Liu, H. Jiang, and S. Chen, Phys. Rev. B **99,** 125103 (2019).

[59] J. Y. Lee, J. Ahn, H. Zhou, and A. Vishwanath, Phys. Rev. Lett. **123,** 206404 (2019).

[60] R. Hamazaki, K. Kawabata, and M. Ueda, Phys. Rev. Lett. **123,** 090603 (2019).

[61] S. Longhi, Phys. Rev. Lett. **122,** 237601 (2019).

[62] B. Höckendorf, A. Alvermann, and H. Fehske, Phys. Rev. Lett. **123,** 190403 (2019).

[63] K. Kawabata, Y. Ashida, H. Katsura, and M. Ueda, Phys. Rev. B **98,** 085116 (2018).

[64] Y. Yi and Z. Yang, Phys. Rev. Lett. **125,** 186802 (2020).

[65] G. Harari, M. A. Bandres, Y. Lumer, M. C. Rechtsman, Y. D. Chong, M. Khajavikhan, D. N. Christodoulides, and M. Segev, Science **359,** eaar4003 (2018).

[66] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Science **359,** eaar4005 (2018).

[67] B. Bahari, A. Ndao, F. Vallini, A. El Amili, Y. Fainman, and B. Kanté, Science **358,** 636 (2017).

[68] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, Phys. Rev. B **99,** 121101(R) (2019).

[69] Y. Michishita and R. Peters, Phys. Rev. Lett. **124,** 196401 (2020).

[70] S. Lieu, M. McGinley, and N. R. Cooper, Phys. Rev. Lett. **124,** 040401 (2020).

[71] K. Kanki, S. Garmon, S. Tanaka, and T. Petrosky, J. Math. Phys. (N.Y.) **58,** 092101 (2017).

[72] W. D. Heiss, J. Phys. A **45,** 444016 (2012).

[73] W. D. Heiss, J. Phys. A **37,** 2455 (2004).

[74] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86,** 787 (2001).

[75] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. **101,** 080402 (2008).

[76] M. Liertzer, L. Ge, A. Cerjan, A. D. Stone, H. E. Türeci, and S. Rotter, Phys. Rev. Lett. **108,** 173901 (2012).

[77] J. Wiersig, Phys. Rev. Lett. **112,** 203901 (2014).

[78] A. Tanaka, S. W. Kim, and T. Cheon, Phys. Rev. E **89,** 042904 (2014).

[79] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, Nature (London) **537,** 76 (2016).

[80] H. Hodaei, Opt. Lett. **41,** 3049 (2016).

[81] M. Kang, J. Chen, and Y. D. Chong, Phys. Rev. A **94,** 033834 (2016).

[82] K.-H. Kim, M.-S. Hwang, H.-R. Kim, J.-H. Choi, Y.-S. No, and H.-G. Park, Nat. Commun. **7,** 13893 (2016).

[83] T. Gao, E. Estrecho, K. Y. Bliokh, T. C. H. Liew, M. D. Fraser, S. Brodbeck, M. Kamp, C. Schneider, S. Höfling, Y. Yamamoto, F. Nori, Y. S. Kivshar, A. G. Truscott, R. G. Dall, and E. A. Ostrovskaya, Nature (London) **526,** 554 (2015).

[84] K. Ding, G. Ma, M. Xiao, Z. Q. Zhang, and C. T. Chan, Phys. Rev. X **6,** 021007 (2016).

[85] C. Shi, M. Dubois, Y. Chen, L. Cheng, H. Ramezani, Y. Wang, and X. Zhang, Nat. Commun. **7,** 11110 (2016).

[86] J. Xu, Y.-X. Du, W. Huang, and D.-W. Zhang, Opt. Express **25,** 15786 (2017).

[87] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Nature (London) **548,** 187 (2017).

[88] A. Pick, B. Zhen, O. D. Miller, C. W. Hsu, F. Hernandez, A. W. Rodriguez, M. Soljačić, and S. G. Johnson, Opt. Express **25,** 12325 (2017).

[89] H. Jing, S. K. Ozdemir, H. Lü, and F. Nori, Sci. Rep. **7,** 3386 (2017).

[90] H. Lü, S. K. Özdemir, L.-M. Kuang, F. Nori, and H. Jing, Phys. Rev. Applied **8,** 044020 (2017).

[91] A. Cerjan, M. Xiao, L. Yuan, and S. Fan, Phys. Rev. B **97,** 075128 (2018).

[92] M. Lyubarov and A. Poddubny, Opt. Lett. **43,** 5917 (2018).

[93] C.-H. Yi, J. Kullig, and J. Wiersig, Phys. Rev. Lett. **120,** 093902 (2018).

[94] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Science **359,** 1009 (2018).

[95] T. Yoshida, R. Peters, and N. Kawakami, Phys. Rev. B **98,** 035141 (2018).

[96] S. Wang, B. Hou, W. Lu, Y. Chen, Z. Q. Zhang, and C. T. Chan, Nat. Commun. **10,** 832 (2019).

[97] W. R. Sweeney, C. W. Hsu, S. Rotter, and A. D. Stone, Phys. Rev. Lett. **122,** 093901 (2019).

[98] A. U. Hassan, B. Zhen, M. Soljačić, M. Khajavikhan, and D. N. Christodoulides, Phys. Rev. Lett. **118,** 093002 (2017).

[99] We note that in some particular examples, e.g., $f_\mu(\boldsymbol{k}) = (\cos k_x - 1) + i(\cos k_y - 1)$, there is only one crossing point formed by a horizontal loop and a vertical loop in the BZ. However, as will be discussed in the following contents, the topological charge of this crossing point is zero, which also preserves the doubling theorem.

[100] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.086401 for the mathematical introduction of the discriminant, the instability of NDPs and high-ordered EPs, the relation between the discriminant number and the vorticity invariant, the connection between a first-ordered EP and a branch cut termination, and examples for the violation of the doubling theorems in the surfaces of the 3D non Hermitians systems.

[101] We note that this procedure can also be applied to find DPs of Hermitian Hamiltonians.

[102] This is because, in a proper basis, all the matrix elements of $\mathcal{H}(\boldsymbol{k})$ are periodic functions of $\boldsymbol{k}$, which is equivalent to the single-valued condition. Therefore, the corresponding characteristic polynomial $f_E(\boldsymbol{k}) = \det[E - \mathcal{H}(\boldsymbol{k})]$, whose coefficients are algebraic functions of these matrix elements, must also be single valued.

[103] Similar to the FPs, it is possible to have single crossing between $\mathrm{Re}[\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})] = 0$ and $\mathrm{Im}[\mathrm{Disc}_E[\mathcal{H}](\boldsymbol{k})] = 0$. However, the topological charge (which will be defined in the following contents) at the crossing point must also be zero, which is unstable to external weak perturbations.

[104] One can easily check that both $F(\boldsymbol{k}) = G(\boldsymbol{k}) = 0$ at (0,0) and $(\pi, 0)$.

[105] Here, we implicitly assume that there is no skin effect, in which case the bulk spectrum would drastically depend on the boundary conditions. Note that in the presence of certain symmetry (e.g., reflection in spinless systems [64]), the skin effect is always absent.

[106] M. M. Denner, A. Skurativska, F. Schindler, M. H. Fischer, R. Thomale, T. Bzdušek, and T. Neupert, arXiv:2008.01090.

[107] C. C. Wojcik, X.-Q. Sun, T. Bzdušek, and S. Fan, Phys. Rev. B **101,** 205417 (2020).